Date - 29/07/2020

B.Sc (Physics) Part-I, Paper-I, Group-B.

Generalized Forces

In earlier class notes we have discussed generalized coordinates and type of constraints on the motion of a particle. Here will will discuss generalized forces..

Generalized coordinates conventionally written as $q_1, q_2, q_3, --- q_n$ or q_k (where $k = 1, 2, 3, --- n$). Here n is the ~~degrees~~ number of degrees of freedom of the system. These q_n coordinates form the configuration space As in the case of Cartesian coordinates, the time derivative of position coordinates gives us velocities, Similarly, derivative of generalized coordinates, i.e., $\dot{q}_1, \dot{q}_2, \dot{q}_3, ---, \dot{q}_n$ or $\dot{q}_k$ may be defined as generalized velocities.

Now ~~let~~ let us consider a single particle whose rectangular coordinates $x, y,$ and z are a function of the generalized coordinates q_1, q_2 and q_3

$$x = x(q_1, q_2, q_3) = x(\vec{q}_k)$$
$$y = y(q_1, q_2, q_3) = y(\vec{q}_k)$$
$$z = z(q_1, q_2, q_3) = z(\vec{q}_k)$$

Suppose the system changes from an initial configuration given by (q_1, q_2, q_3) to a neighborhood configuration given by $(q_1 + \delta q_1, q_2 + \delta q_2, q_3 + \delta q_3)$. We can express the corresponding changes in the coordinates by the relation

$$\delta x = \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2 + \frac{\partial x}{\partial q_3}\delta q_3 = \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\delta q_k$$

with similar expression for δy and δz; where n is equal to three and the partial derivatives $\frac{\partial x}{\partial q_3}$ etc. are functions of q's.

The generalization of above equation –

Consider a mechanical system having large number of particles and n degrees of freedom. The configuration of the system is specified by coordinates $q_1, q_2, \ldots q_n$. Let the configuration changes from $(q_1, q_2, \ldots, q_n)$ to a new configuration $(q_1 + \delta q_1, q_2 + \delta q_2, \ldots, q_n + \delta q_n)$. The Cartesian coordinates of a particle i changes from (x_i, y_i, z_i) to $(x_i + \delta x_i, y_i + \delta y_i, z_i + \delta z_i)$.

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\delta q_1 + \frac{\partial x_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial x_i}{\partial q_n}\delta q_n$$

$$= \sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k}\delta q_k$$

~~Same for~~

Similarly δy_i and δz_i can be written.

Actual & virtual displacement -

Let actual displacement is denoted by $\rightarrow d\vec{r_i}$

Virtual displacement $\rightarrow \delta\vec{r_i}$

Let a mass m_i is acted on by an external force $\vec{F_i}$ and causes the mass m_i to move from $\vec{r_i}$ to $\vec{r_i} + d\vec{r_i}$ in time interval dt. This displacement must be consistent with both the equations of motion and the equations of constraints that describes this mass system; hence such displacements are actual displacement. On the other hand, virtual displacements are consistent with the equations of the constraints but do not ~~stisify~~ satisfy equations of motion or time.

Example: The bob of a pendulum of length l may be moved from (l, θ) to $(l, \theta + \delta\theta)$ in any arbitrary time interval as long as the bob remains on the arc of a circle of radius l. Thus $\delta\vec{r_i}$ and δq_i are virtual displacements.

We will define the concept of virtual work.

Virtual work:-

Consider a force $\vec{F}$ that is acting on a single particle of mass m and produces a virtual displacement $\delta\vec{r}$ of the particle. The work done δW by the force is given by

$$\delta W = \vec{F} \cdot \delta\vec{r} = F_x \delta x + F_y \delta y + F_z \delta z.$$

here F_x, F_y, F_z are the rectangular components of $\vec{F}$.

We can also write δW as

$$\delta W = \sum_{k=1}^{n} \left(F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k} + F_z \frac{\partial z}{\partial q_k} \right) = \sum_{k=1}^{n} Q_k \, \delta q_k$$

where $Q_k = F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k} + F_z \frac{\partial z}{\partial q_k}$

Q_k is generalized force associated with the generalized coordinate q_k. Dimension of Q_k depend on the dimension of q_k. Dimension of $Q_k \, \delta q_k$ is ~~work~~ dimension of work. If δq_k has dimension of distance, Q_k will have dimension of force. If δq_k has dimension of angle θ, Q_k will have dimension of torque T_θ. It is to be noted that δq_k and $\delta x, \delta y, \delta z$ are virtual displacements of the system because it is not necessary that these displacements represent the actual displacements.

Generalize to N particle system –

$$\delta W = \sum_{i=1}^{N} \vec{F}_i \cdot \delta \vec{r}_i = \sum_{i=1}^{N} \left[F_{x_i} \delta x_i + F_{y_i} \delta y_i + F_{z_i} \delta z_i \right]$$

$$\delta W = \sum_{i=1}^{N} \left\{ \sum_{k=1}^{n} \left(F_{x_i} \frac{\partial x_i}{\partial q_k} + F_{y_i} \frac{\partial y_i}{\partial q_k} + F_{z_i} \frac{\partial z_i}{\partial q_k} \right) \delta q_k \right\}$$

or $$\delta W = \sum_{k=1}^{n} \left\{ \sum_{i=1}^{N} \left(F_{x_i} \frac{\partial x_i}{\partial q_k} + F_{y_i} \frac{\partial y_i}{\partial q_k} + F_{z_i} \frac{\partial z_i}{\partial q_k} \right) \delta q_k \right\}$$

or $\delta W = \sum_{k=1}^{n} Q_k \, \delta q_k$, where $Q_k = \sum_{i=1}^{N} \left(F_{x_i} \frac{\partial x_i}{\partial q_k} + F_{y_i} \frac{\partial y_i}{\partial q_k} + F_{z_i} \frac{\partial z_i}{\partial q_k} \right)$

Generalized force.